A Batch Ballot Theorem and its Application to M/G/1 Type Queues

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1. Introduction

Ballot Theorems are used extensively in a variety of applications in the area of stochastic processes. Takács, [?] [?] and [?] have illustrated application of the Ballot theorem and its generalisations. The application of the generalised ballot theorem to queuing theory leads to elegant results for the simple $M/G/1$ queue. It is thought that such results are not possible for more general $M/G/1$ type queues. We, however, derive a batch ballot theorem which can be applied to derive the first passage distribution matrix, $G$, for the general $M/G/1$ type queue.

2. Batch Ballot Theorem

In this section we develop a general Batch Ballot theorem. First we state the generalised Ballot theorem of Takács proved in [?].

**Theorem 1 (Takács(1967))** Let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 + n_2 + \ldots + n_k = n \leq k$. Among the $k$ cyclic permutations of $(n_1, n_2, \ldots, n_k)$ there are exactly $k - n$ for which the sum of the first $s$ elements is less than $s$ for all $s = 1, 2, \ldots, k$.

Mendelson, [?], considered counting the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first $r$ elements is less than $rm$ for $r = 1, 2, \ldots, k$. For the extended sequence $n_1, n_2, \ldots, n_k, n_{k+1}, \ldots$, where $n_{k+r} = n_r$ for $r \geq 1$, Mendelson defined $\delta_r$ and $\Psi_r$ by $\delta_r = 1$, if $jm - \phi_j > rm - \phi_r$ for $j > r$, and $= 0$ otherwise; $\Psi_r = \inf_{i \geq r} \{jm - \phi_j\}$, where $\phi_r = \sum_{j=1}^r n_j$. For the case where $m = 1$, Takács in [?] notes that $\Psi_{r+1} - \Psi_r = \delta_r$ and further proves that the number of cyclic permutations satisfying the criteria of Theorem 1 is equal to $\sum_{r=1}^k \delta_r = \sum_{r=1}^k (\Psi_{r+1} - \Psi_r) = k - n$. Mendelson in [?] was able to show, for $m > 1$, that $\Psi_{r+1} - \Psi_r$ may exceed unity and that $\delta_r \leq \Psi_{r+1} - \Psi_r$. As a consequence, the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first $s$ elements is less than $sm$ for $s = 1, 2, \ldots, k$ is bounded by $\sum_{r=1}^k \delta_r = \sum_{r=1}^k (\Psi_{r+1} - \Psi_r) \leq km - n$. In the following lemma and theorems we provide exact results for these quantities. Details of the proofs
are given in [?]. Here $\phi_t$ and $\Psi_r$ are as before but, for $1 \leq r \leq k$, redefine $\delta_r$ as

$$\delta_r = \begin{cases} 
  m & \text{if } jm - \phi_j > rm - \phi_r + m - 1 \text{ for } j > r, \\
  d & \text{if } jm - \phi_j > rm - \phi_r + d - 1 \text{ for } j > r \text{ and } um - \phi_u = rm - \phi_r + d \\
  0 & \text{otherwise}
\end{cases}$$

Theorem 2 Let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 + n_2 + \ldots + n_k = n < km$. Then

(a) $0 \leq \delta_r = \Psi_{r+1} - \Psi_r \leq m$.

(b) $\sum_{r=1}^{k} \delta_r = \sum_{r=1}^{k} (\Psi_{r+1} - \Psi_r) = km - n$.

(c) For $0 \leq d \leq m - 1$, let $C_d$ denote the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first $s$ elements is less than $sm - d$ for $1 \leq s \leq k$. Then $C_0 + C_1 + \ldots + C_{m-1} = km - n$.

As a consequence of the above result we have following theorem.

Theorem 3 Let $\nu_1, \ldots, \nu_k$ be cyclically interchangeable random variables taking on nonnegative integral values. Set $N_s = \nu_1 + \ldots + \nu_s$ for $1 \leq s \leq k$ with $N_0 = 0$. Then, for $0 \leq n \leq km$, we have

$$\sum_{d=0}^{m-1} \Pr[N_s < sm - d \text{ for } 1 \leq s \leq k | N_k = n] = \frac{(km - n)}{k}.$$

3. Application to M/G/1 Type Queues

Queues of M/G/1 type arise extensively in the fields of teletraffic analysis and engineering. One of the main quantities of interest for such queues is the first passage distribution matrix $G$. Several methods are developed for the evaluation of $G$. These are mainly based upon implementing successive substitutions on a truncated form of a non-linear matrix equation ([?]). Takács, [?] and [?], applied a generalised Ballot theorem to the simple M/G/1 queue and obtained elegant results. It has been long thought, [?], that such results were not possible for the general M/G/1 type queue. We, however, apply the batch ballot theorem derived in the previous section to compute the matrix $G$ for the general M/G/1 type queue. Details of this are given in [?].

REFERENCES


