A test for a conjunction

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A conjunction is defined in the brain mapping literature as the occurrence of the same event at the same location in two or more independent 3D brain images (see Figure 1). The images are smooth isotropic 3D random fields of test statistics, and the event occurs when the image exceeds a fixed high threshold. We give a simple approximation to the probability of a conjunction occurring anywhere in a fixed region, so that we can test for a local increase in mean of the images at the same unknown location in all images, a generalization of the split-t test. This is the corollary to a more general result on the expected Minkowski functionals of the set of points where a conjunction occurs.

Let $X_i(\mathbf{t})$ be the value of image i at location $\mathbf{t} \in \Re^D$, $1 \le i \le n$, and let x be a fixed threshold. The excursion set A_i is the set of points where the ith image exceeds the threshold: $A_i = \{\mathbf{t} \; : \; X_i(\mathbf{t}) \ge x\}$. The conjunction C is where all the images exceed the threshold, that is, the intersection of the excursion sets, intersected with a fixed search region $S \subset \Re^D$:

$$C = \{\mathbf{t} \in S : X_i(\mathbf{t}) \ge x \text{ for all } 1 \le i \le n\} = A_1 \cap \cdots \cap A_n \cap S.$$

An example is shown in Figure 1 for $n = 6$, where S is the whole brain. We are interested in the probability that C is not empty, that is, the probability that all images exceed the threshold at some point inside S, or that the maximum over $\mathbf{t} \in S$ of the minimum over i of $X_i(\mathbf{t})$ exceeds x:

$$\mathcal{P}\{C \ne \emptyset\} = \mathcal{P}\left\{\max_{\mathbf{t}\in S} \min_{1\le i\le n} X_i(\mathbf{t}) \ge x\right\}. \tag{1}$$

If the images are independent and identically distributed stationary random fields then the expected Lebesgue measure or volume of C is

$$\mathcal{E}\{|C|\} = p^n|S|, \tag{2}$$

where $p = \mathcal{P}\{X_i(\mathbf{t}) \ge x\}$. Our main result (7) is that (2) holds if Lebesgue measure is replaced by a vector of Minkowski functionals (also called intrinsic volumes), and p is replaced by a matrix of Euler characteristic intensity functions for the random field. This gives (2) as a special case, and other interesting quantities such as the expected surface area of C, which comes from the $D - 1$ dimensional Minkowski functional. But the component of most interest to us is the zero dimensional Minkowski functional, or Euler characteristic (EC). For high thresholds, the expected EC of C is a very accurate approximation to the probability (1) that we seek (Adler, 2000).

This also allows us to set the level of the split-t test. Shaywitz *et al.* (1995) used this test to determine whether the functional organization of the brain for language differed according to sex. 38 independent fMRI images were randomly divided into $n = 2$ groups, and an image $X_i(\mathbf{t})$ of t-statistics was calculated for each group $i = 1, 2$. The split-t test rejects if the null hypothesis is rejected for both groups at the same point $\mathbf{t}$, that is, if $X_1(\mathbf{t}) \ge x$ and $X_2(\mathbf{t}) \ge x$ for some threshold x, taken as the upper level 5% point of the t-distribution with 17 degrees of freedom. The resulting image of conjunctions C appears on the cover of *Nature* that contains Shaywitz *et al.* (1995).

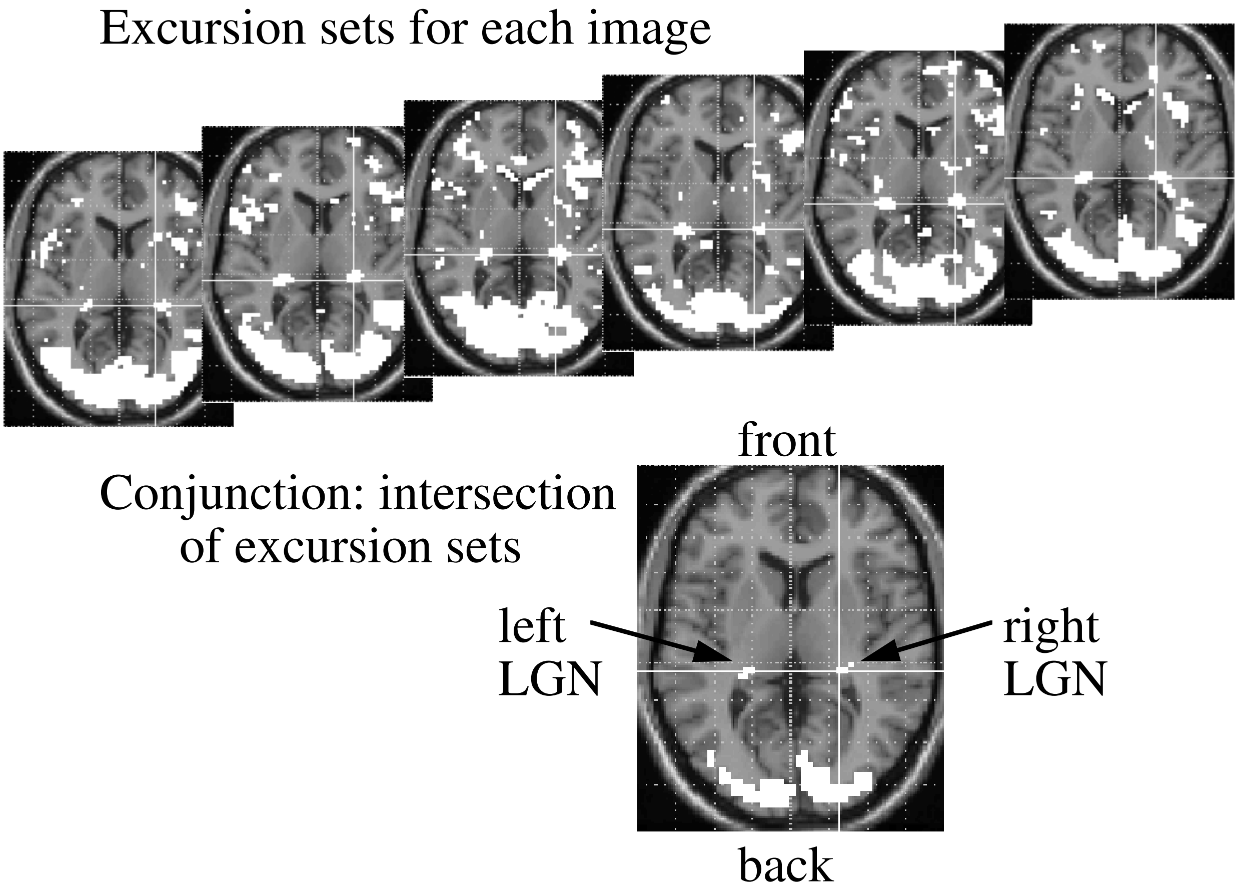

Figure 1: Conjunction of $n = 6$ fMRI images during a visual task (only one slice of the 3D data is shown). The excursion sets where $X_i(\mathbf{t}) \geq 1.64$, $i = 1, \ldots, 6$ are shown in white on a background of brain anatomy (top). The set of conjunctions C is the intersection of these sets and the brain, S (bottom). The visual cortex at the back of the brain appears in C, but the most interesting feature is the appearance of the lateral geniculate nuclei (LGN) (arrows).

Let $\mu_i(A)$ be the ith Minkowski functional of a set $A \subset \Re^D$, scaled so that it is invariant under embedding of A into any higher dimensional Euclidean space. If A has a twice differentiable boundary ∂A, then it can be defined as follows. Let $s_i = 2\pi^{i/2}/\Gamma(i/2)$ be the surface area of a unit $(i-1)$-sphere in $\Re^i$. For $\mathbf{M}$ an $m \times m$ matrix let $\mathrm{detr}_j(\mathbf{M})$ denote the sum of the determinant of all $j \times j$ principal minors of $\mathbf{M}$, so that $\mathrm{detr}_m(\mathbf{M}) = \det(\mathbf{M})$, $\mathrm{detr}_1(\mathbf{M}) = \mathrm{tr}(\mathbf{M})$ and we define $\mathrm{detr}_0(\mathbf{M}) = 1$. Let $\mathbf{Q}$ be the $(D-1) \times (D-1)$ curvature matrix of ∂A. Then for $0 \leq i < D$

$$\mu_i(S) = \frac{1}{s_{D-i}} \int_{\partial A} \mathrm{detr}_{D-1-i}(\mathbf{Q}) d\mathbf{t},$$

and define $\mu_D(A) = |A|$. Note that $\mu_0(A)$ is the EC of A by the Gauss-Bonnet Theorem, and $\mu_{D-1}(A)$ is half the surface area of A. For example, the Minkowski functionals of a ball of radius r in $\Re^3$ are

$$\mu_0(S) = 1, \quad \mu_1(S) = 4\pi r, \quad \mu_2(S) = 2\pi r^2, \quad \mu_3(S) = (4/3)\pi r^3. \qquad (3)$$

We shall use the result that any set functional $\psi(A)$ that obeys the additivity rule

$$\psi(A \cup B) = \psi(A) + \psi(B) - \psi(A \cap B) \tag{4}$$

is a linear combination of the Minkowski functionals. If $X(\mathbf{t})$, $\mathbf{t} \in \Re^D$, is an isotropic random field with excursion set $A = \{\mathbf{t} : X(\mathbf{t}) \geq x\}$ then

$$\mathcal{E}\{\mu_0(A \cap S)\} = \sum_{i=0}^{D} \rho_i \mu_i(S) \tag{5}$$

for some constants ρ_i. This follows from the fact that $\psi(S) = \mathcal{E}\{\mu_0(S \cap A)\}$ obeys the additivity rule (4), since μ_0 does, so it must be a linear combination of the Minkowski functionals $\mu_i(S)$. The coefficients ρ_i, called Euler characteristic (EC) intensities in $\Re^i$, can be evaluated for a variety of random fields (Adler, 1981; Worsley, 1994, 2001; Siegmund & Worsley, 1995; Cao & Worsley, 1999ab). For example, for a Gaussian random field with $\mathcal{E}\{X(\mathbf{t})\} = 0$, $\mathcal{V}ar\{X(\mathbf{t})\} = 0$, $\mathcal{V}ar\{\partial X(\mathbf{t})/\partial \mathbf{t}\} = \lambda \mathbf{I}$, where $\mathbf{I}$ is the $D \times D$ identity matrix, then $\rho_0 = \mathcal{P}\{X(\mathbf{t}) \geq x\}$ and for $i > 0$

$$\rho_i = \lambda^{i/2}(2\pi)^{-(i+1)/2}\mathrm{He}_{i-1}(x)e^{-x^2/2}, \tag{6}$$

where $\mathrm{He}_j(x)$ is the Hermite polynomial of degree j in x.

Our main result is an expression for the expected Minkowski functionals of the conjunction. For $B \subset \Re^D$, let $\mu(B) = (\mu_0(B), \mu_1(B), \ldots, \mu_D(B))'$ be the column vector of Minkowski functionals of B. Define the $D+1 \times D+1$ upper triangular matrix $\mathbf{P}$ with ijth element

$$\rho_{j-i} \frac{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{j-i+1}{2}\right)}$$

if $j \geq i$ and 0 otherwise. Then the expected Minkowski functionals of the conjunction are

$$\mathcal{E}\{\mu(C)\} = \mathbf{P}^n \mu(S). \tag{7}$$

The proof of this result is given in Worsley & Friston (2000).

Comparing (7) with (2) we see that it has the same form, with volume replaced by the vector of weighted Minkowski functionals, and probability replaced by the matrix of weighted EC intensities. The last element is the same as (2), and the first element is the expected EC of the set of conjunctions that we shall use as an approximation to the probability of a conjunction anywhere in S:

$$\mathcal{P}\{C \neq \emptyset\} \approx (1, 0, \ldots, 0)\mathbf{P}^n \mu(S), \tag{8}$$

for high thresholds x.

We shall apply (8) to some $D = 3$ dimensional functional magnetic resonance imaging (fMRI) data fully described in Friston *et al.* (1999). The purpose of the experiment was to determine those regions of the brain that were consistently stimulated by all subjects while viewing a pattern of radially moving dots. To do this, subjects were presented with a pattern of moving dots, followed by a pattern of stationary dots, and this was repeated 10 times, during which a total of 120 3D fMRI images were obtained at the rate of one every 3.22 seconds. For each subject i and at every point $\mathbf{t} \in \Re^3$, a test statistic $X_i(\mathbf{t})$ was calculated for comparing the fMRI response between the moving dots and the stationary dots. Under the null hypothesis of no difference, $X_i(\mathbf{t})$ was modeled as an isotropic Gaussian random field with zero mean, unit variance and $\lambda = 4.68\text{cm}^{-2}$. A threshold of $x = 1.64$, corresponding to an uncorrected level 5% test, was chosen, and the excursion sets for each subject are shown in Figure 1, together with their intersection, which forms the set of conjunctions C. The search region S is the whole brain area that was scanned, which was an approximate spherical region with a volume of $|S| = 1226\text{cm}^3$. Finally, the approximate probability of a conjunction, calculated from (3), (6) and (8), is 0.0126. We can thus conclude, at the 1.26% level, that conjunctions have occurred in the visual cortex, and more interesting, the lateral geniculate nuclei (see Figure 1).

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