General Stochastic Differential Equation Models of Population Growth and Fishing in a Random Environment

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1. Introduction and models

Let $N=N(t)$ be the population size at time $t\geq 0$ and $N(0)=N_0>0$. A general deterministic density-dependent population growth model without Allee effects puts the *growth effort* (*per capita growth rate*) $(1/N)dN/dt=g(N)$, where g is a class C^2 real function defined for $N>0$. Let $G(N)=Ng(N)$ be the *total growth rate.* Assume $g(0^+)>0$ (small populations grow), $g(+\infty)<0$ (large populations decrease), $dg(N)/dN<0$ (the larger the population, the harder it is for an individual to survive and reproduce), $G(0^+)=0$ (no spontaneous generation). This implies that there is a unique $K>0$, called *carrying capacity,* such that $g(K)=0$; it is the unique stable equilibrium and $N(t)\to K$ as $t\to+\infty$.

In an environment subjected to random fluctuations, we may approximate them by a standard white noise $e(t)$ multiplied by a (*per capita*) *noise intensity* $s(N)$ (a class C^2 strictly positive function defined for $N>0$). Let $S(N)=Ns(N)$ be the *total noise intensity.* Assume that $S(0^+)=0$ (to avoid negative populations or spontaneous generation). Assume also: (A) $s(N)=O(g(N))$ when $N\to 0^+$ and when $N\to+\infty$; (B) $\int_{0^+}^{x_0}(1/S(N))dN=\int_{x_0}^{+\infty}(1/S(N))dN=+\infty$ for some $x_0>0$. A sufficient condition for (A) and (B) to hold is that the *per capita* noise intensity $s(N)$ be bounded. We obtain the *Stratonovich stochastic differential equation* (SDE) model

(1) $$(1/N)dN/dt=g(N)+s(N)e(t).$$

If there is fishing with a density-dependent *fishing effort* $f(N)$, we assume it to be a non-negative class C^2 function defined for $N>0$ such that the *yield* (total fishing rate) $F(N)=Nf(N)$ satisfies $F(0^+)=0$ (one can not fish if there is no fish). Let $h(N)=f(N)/g(N)$. With fishing, the model is

(2) $$(1/N)dN/dt=g(N)-f(N)+s(N)e(t).$$

Although modelers assume g to be of some specific form (e.g. logistic, Gompertz), g is really poorly known. Fishing policies f can also be quite variable. So it is important to obtain results that are independent of the specific form of these functions. That is the reason we have consider g and f arbitrary, satisfying only mild assumptions dictated by biology.

2. Results

Models of types (1) and (2) (sometimes using Ito instead of Stratonovich calculus) have been proposed in the literature for very specific forms of the functions g, f and s (see references in Braumann 1999a, 1999b). Usually, results on non-extinction and on existence of a stationary density are obtained. Existence of a stationary density is important because it implies that, although population size keeps varying randomly over time, its probability distribution stabilizes (converges to the stationary distribution as $t\to+\infty$).

Braumann (1999a, 1999b), resp. for models (1) and (2), generalizes these results to general functions g and f satisfying the mild assumptions referred to above, but assuming the noise intensity s to be constant or proportional to g. The case of s proportional to g will not be considered here because it implies an unrealistic zero noise intensity at the carrying capacity. For model (2), one assumes also $h(0^+)<1$ (the fishing effort should be smaller than the growth effort at small population sizes); the case $h(0^+)>1$ would imply “mathematical” extinction ($N \to 0$ as $t \to +\infty$) a.s.

In the case $s>0$ constant, those papers prove that the SDE (1) and (2) have unique solutions, which are diffusion processes with diffusion coefficient $b(N)=S^2(N)$ and drift coefficient $a(N)= G(N)-F(N)+(1/4)(db(N)/dN)$ (of course, for model (2), $F(N)\equiv 0$), that “mathematical” extinction has zero probability of occurring, and that there is a stationary density

$$(3) \qquad p(N)=\frac{m(N)}{\int_0^{+\infty} m(x)dx} \ (0<N<+\infty), \text{ with } m(N)=\frac{1}{b(N)}\exp\left(\int_{x_0}^{N}\frac{2a(z)}{b(z)}dz\right) \ (x_0>0 \text{ arbitrary}).$$

Also, for small s, the modes [antimodes] of the stationary distribution approximately coincide with the stable [unstable] equilibria of the corresponding deterministic model (the model with $s=0$).

In Braumann (2001a, 2001b) these results were further generalized to an arbitrary density-dependent noise intensity $s(N)$ satisfying the mild assumptions referred to above.

Due to the continuity of the solution of the SDE (1) and (2), one can infer some global properties from the behavior at the boundaries $N=0$ and $N=+\infty$. Using results that can be seen, for instance, in Karlin and Taylor (1981) and Gihman and Skorohod (1972), the proofs can be reduced to proving that the boundaries are non-attracting (to show non-explosion and non-extinction of the solution) and that the speed density $m(x)$ is integrable in $(0,+\infty)$ (to show existence of a stationary density), which can be done, with a bit of work, using the mild assumptions above.

For parameter estimation, see Braumann (1997) and references therein.

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RESUMÉ

Nous présentons résultats concernant la non-extinction et l’existence de densité stationaire pour le grandeur des populations croissant dans un environnement avec des fluctuations aléatoires, soit elles soumis à la pêche ou non. Les résultats sont assez indépendants des modèles (équations différentielles stochastiques), parce qu’ils sont très généraux.