Empty confidence sets for epidemics and branching processes

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1. Introduction

This paper outlines some results, which will be discussed in more detail in Ball *et al* (2001), concerning the construction of confidence sets for examples of probability models which display a threshold phenomenon. In each case, for certain observations, the confidence set for the parameter of interest is empty. This happens when the obserservation is extremely unlikely for all choices of model parameter. In practical terms it seems sensible to interpret an empty confidence set as an indication that the model is inappropriate for the data.

2. Epidemics

Consider the following model for the spread of an epidemic among a population comprised initially of a infectives and n susceptibles. The infectious periods of different infectives are independently and identically distributed according to a random variable T_I, having an arbitrary but specified distribution. Throughout its infectious period, a given infective makes contacts at the points of a Poisson process with rate $n\beta$. Successive contacts are with individuals chosen independently and uniformly from the n initial susceptibles. If a contacted individual is still susceptible then it becomes infected, otherwise nothing happens.

The threshold behaviour of this epidemic model is well known. Let T denote the number of initial susceptibles that are ultimately infected and $R_0 = n\beta E[T_I]$. When $R_0 \leq 1$, the distribution of T is unimodal with the mode at zero, so only minor epidemics occur. When $R_0 > 1$, the distribution of T still has a mode at zero but there is a second mode, corresponding to the occurence of a major epidemic.

The total size T is stochastically increasing in R_0, so a natural method for constructing an equal-tailed confidence interval for R_0, from an observation, t_{OBS} say, of T, is by inverting the family of hypothesis tests of $H_0 : R_0 = R_0'$ versus $H_1 : R_0 \neq R_0'$ which reject H_0 if t_{OBS} is too extreme. However, the resulting confidence intervals are very wide, owing to the distribution of T being bimodal when $R_0 > 1$, and their upper limit is the same for most values of t_{OBS}, a consequence of the threshold behaviour of the epidemic. Thus *inverted-likelihood-based* confidence sets, derived by inverting the family of hypothesis tests of $H_0 : R_0 = R_0'$ versus $H_1 : R_0 \neq R_0'$ having acceptance region containing the most likely values of T under H_0, are considered. Numerical calculations show that these confidence sets do not have the above undesirable features, although they may not be single intervals and are empty for some values of t_{OBS} if n is sufficiently large.

3. Branching processes

Consider a Galton-Watson branching process with a initial ancestors, in which the number of offspring of a single individual is distributed according to a random variable Z_λ, where $\lambda = E[Z_\lambda]$. Let T_λ denote the total progeny of the branching process and, for $k = 0, 1, \ldots$, define $p_k(\lambda) = P(T_\lambda = k)$. Suppose that the following three conditions hold: (i) for fixed $\lambda > 0$, $p_k(\lambda) \geq p_{k+1}(\lambda)$ $(k = 0, 1, \ldots)$; (ii) $T_\lambda \geq_{st} T_{\lambda'}$ for any $\lambda > \lambda'$; and (iii) for any non-negative integer n, $P(T_\lambda \in [n, \infty))$ decreases with λ for $\lambda \geq 1$. Then, if the value $T_\lambda = t$ is observed, the inverted-likelihood-based $100(1-\alpha)\%$ confidence set for λ takes the form

$$R_\alpha(t) = \begin{cases} [\lambda_\alpha^l(t), \lambda_\alpha^u(t)] & \text{if } t \leq b(\alpha), \\ \emptyset & \text{if } b(\alpha) < t < \infty, \\ [\lambda_\alpha^*, \infty) & \text{if } t = \infty, \end{cases}$$

where, for fixed α, $\lambda_\alpha^l(t)$ increases and $\lambda_\alpha^u(t)$ decreases with t. Examples of offspring distributions for which conditions (i) to (iii) hold (for sufficiently small a) include geoemtric, Poisson, binomial and negative binomial.

REFERENCES

Ball, F.G., Britton, T. and O'Neill, P.D. (2001) Empty confidence sets for epidemics, branching processes and Brownian motion. In preparation.

RESUME

Dans cet article, on construit des ensembles de confiance, d'une part pour le nombre de reproduction de base d'une épidémie, en se fondant sur l'observation de la taille totale de l'épidémie, et d'autre part pour la moyenne de la distribution du nombre de descendants directs dans un processus de Galton-Watson, en se fondant sur l'observation de la descendance totale de ce processus. Dans les deux cas, les ensembles de confiance sont vides pour certaines valeurs observées.